

# Macroscopic Quantum Tunnelling in Rotating Bose-Einstein Condensates

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## Abstract

In this paper we investigate the macroscopic quantum tunnelling and the phase coherence property of the rotating Bose-Einstein condensates in both static and dynamic cases by using the mean field theory.

## I. INTRODUCTION

It is well known that the phase coherence and the macroscopic quantum phenomenon are among the important characteristics of the Bose-Einstein condensates ( BEC ), and have induced a lot of theoretical and experimental interests. Now, we came to know that the phase coherence property and the macroscopic tunnelling are so closely related that such a system was found to exhibit quite rich physics. On the other hand, motivated by the superfluidity of the dilute Bose gas, much work are focused on the rotating BEC recently [1]. In this paper, we investigate the macroscopic quantum tunnelling in the rotating BEC in various parameter ranges and in both static and dynamic cases with the help of the mean field approach, and the phase coherence property in this system will be emphasized.

In Section II, a general description of our system and the Hamiltonian to begin with are given. In the next two sections, we focus on the static property when the rotating frequency is fixed. SU(2) coherent state path integral method is used in Section III and we find that if the parameters satisfy a specific condition, the ground state represents the superposition of two macroscopic quantum states. The energy splitting due to the macroscopic coherent tunnelling is also calculated by the instanton technique. In another parameter range, fluctuations of the phase and the particle number of the ground state are given in Section IV and it follows that the condition of the Josephson effect in this case is satisfied. In Section V, we consider the case where the rotating frequency varies with time, and the nonlinear Landau-Zener tunnelling [2] is realized in this system. Finally, some conclusions and further discussion are given in Section VI.

## II. MODEL

We consider the Bose-Einstein condensate with total particle number  $N$  which is trapped in a circularly symmetrical trap  $V$  or container, and rotated by an asymmetric perturbation  $V_{ext}$ . The N-body Hamiltonian in the rotating frame of such a system is given by  $H = \sum_{n=1}^N [H_{0n} + V_{ext}(\vec{r}_n) - \omega \cdot L_{zn}] + \sum_{1 \leq i < j \leq N} U \delta(\vec{r}_i - \vec{r}_j)$ , where  $H_{0n} = \vec{p}_n^2/2m + V(\vec{r}_n)$  is the single body Hamiltonian in the laboratory frame,  $U = 4\pi a \hbar^2/m$ ,  $a$  is the s-wave scattering length and  $a > 0$  for repulsive interactions,  $\omega$  is the angular frequency of the rotating frame. The lowest-lying S and P states of the Hamiltonian  $H_0$  are separated by an energy  $\hbar\omega_0$ .

Following [1], we adopt the two-mode approximation, i.e. under the condition that the effective interacting energy  $Un$  is much smaller than  $\hbar\omega_0$  and the energy of the first radial excitation is much large than  $\hbar\omega_0$ , where  $n$  is the particle number density, we expand the field operator in terms of the single particle wave functions of the two lowest-lying states which are exact eigenstates of  $L_z$ ,  $\Psi(\vec{r}) = a_1\varphi_1(\vec{r}) + a_0\varphi_0(\vec{r})$ . Following the conventional procedure and omitting constant terms proportional to  $N$ , we get the second quantization Hamiltonian

$$H = \hbar\delta\omega(a_0^\dagger a_0 - a_1^\dagger a_1) - V_0(a_0^\dagger a_1 + a_1^\dagger a_0) + \frac{U}{2}(\tilde{g}_0 a_0^\dagger a_0^\dagger a_0 a_0 + \tilde{g}_1 a_1^\dagger a_1^\dagger a_1 a_1 + 4\tilde{g}_{01} a_0^\dagger a_0 a_1^\dagger a_1), \quad (1)$$

where  $\delta\omega = \frac{1}{2}(\omega - \omega_0)$ , the coupling energy  $V_0 \equiv -\int \varphi_1^*(\vec{r}) V_{ext} \varphi_0(\vec{r}) d^3r$  is set real and positive by fixing a proper phase difference between  $\varphi_0$  and  $\varphi_1$  and it can be adjusted by changing the strength of the perturbation. The  $\tilde{g}$ 's are determined by the eigenfunctions  $\varphi_0$  and  $\varphi_1$  in the following way  $\tilde{g}_0 \equiv \int |\varphi_0(\vec{r})|^4 d^3r$ ,  $\tilde{g}_1 \equiv \int |\varphi_1(\vec{r})|^4 d^3r$ ,  $\tilde{g}_{01} \equiv \int |\varphi_0(\vec{r})|^2 |\varphi_1(\vec{r})|^2 d^3r$ .

### III. MACROSCOPIC QUANTUM COHERENT TUNNELLING

With the help of the mapping

$$\begin{aligned} \frac{1}{2}(a_0^\dagger a_0 - a_1^\dagger a_1) &\rightarrow J_z, \\ \frac{1}{2}(a_0^\dagger a_1 + a_1^\dagger a_0) &\rightarrow J_x, \\ -\frac{1}{2}i(a_0^\dagger a_1 - a_1^\dagger a_0) &\rightarrow J_y, \\ \frac{1}{2}(a_0^\dagger a_0 + a_1^\dagger a_1) &\rightarrow J, \end{aligned} \quad (2)$$

our problem is reduced to the one of a particle with large spin  $J$  in an arbitrary  $\vec{J}$ -dependent potential, i.e. the Hamiltonian of the system can be rewritten as when dropping lower terms in self-interactions

$$H = \hbar\delta\omega J_z - V_0 J_x + \frac{U}{2}[\tilde{g}_0(J + J_z)^2 + \tilde{g}_1(J - J_z)^2 + 4\tilde{g}_{01}(J^2 - J_z^2)] \quad (3)$$

First of all, we introduce the SU(2) coherent states [3]

$$\begin{aligned} |\Omega\rangle &= |\theta, \phi\rangle = e^{(\zeta J_+ - \zeta^* J_-)} \left| \frac{N}{2}, -\frac{N}{2} \right\rangle \\ &= \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \sqrt{\frac{N!}{(\frac{N}{2} + n)! (\frac{N}{2} - n)!}} \left( \cos \frac{\theta}{2} \right)^{\frac{N}{2} - n} [\exp(-i\phi) \sin \frac{\theta}{2}]^{\frac{N}{2} + n} |n\rangle. \end{aligned} \quad (4)$$

Here,

$$\zeta = \frac{\theta}{2} e^{-i\phi} \quad (0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi), \quad (5)$$

$|\frac{N}{2}, -\frac{N}{2}\rangle$  is the extremal state  $|J = \frac{N}{2}, J_z = -\frac{N}{2}\rangle$ ,  $|n\rangle$  is the eigenstate of  $J_z$ ,  $2n$  represents the occupation number difference between the two states and  $\phi$  stands for the phase difference between them which will be given explicitly in the next section.  $SU(2)$  coherent states are overcomplete in the Hilbert space and any wave function can be expanded in terms of them, for example, a Fock state  $|n\rangle$  can be written as

$$|n\rangle \propto \int_0^{2\pi} d\phi e^{i(n+N/2)\phi} |\Omega\rangle. \quad (6)$$

In the  $SU(2)$ -coherent-state path-integral representation, the Euclidean transition amplitude from an initial state to a final state is [3]

$$\langle \Omega_f | e^{-H(\tau_f - \tau_i)/\hbar} | \Omega_i \rangle = \int [d\Omega(\tau)] \exp \left[ -\frac{1}{\hbar} S_E(\theta, \phi) \right], \quad (7)$$

where

$$S_E(\theta, \phi) = \int_{\tau_i}^{\tau_f} d\tau \left[ i\hbar \frac{N}{2} (1 - \cos \theta) \left( \frac{d\theta}{d\tau} \right) + E(\theta, \phi) \right]. \quad (8)$$

The first term in Eq. (8) is the Wess-Zumino term, and the semiclassical energy  $E(\theta, \phi)$  accurate to a constant proportional to  $N$  is given by

$$\frac{1}{N} E(\theta, \phi) = [\hbar\delta\omega + \frac{1}{4}(g_0 - g_1)] \cos \theta - V_0 \sin \theta \cos \phi + \frac{1}{2}[g_{01} - \frac{1}{4}(g_0 + g_1)] \sin^2 \theta, \quad (9)$$

where

$$\begin{aligned} g_0 &= U\tilde{g}_0 N, \\ g_1 &= U\tilde{g}_1 N, \\ g_{01} &= U\tilde{g}_{01} N. \end{aligned} \quad (10)$$

Since  $\sin \theta \geq 0$  ( $0 \leq \theta \leq \pi$ ), for  $V_0 > 0$  the minima of the semiclassical energy can only occur for  $\phi = 0$ .

It turns out that the qualitative behavior of  $E(\theta, \phi)$  depends on a relation between  $V_0$  and the  $g$ 's. The first regime is characterized by the presence of two minima. For  $\delta\omega \neq \frac{1}{4\hbar}(g_1 - g_0)$  the minima define a ground state and a metastable state. For  $\delta\omega = \frac{1}{4\hbar}(g_1 - g_0)$  the two minima are degenerate, and a situation of macroscopic quantum coherence occurs. In the other regime  $E$  as a function of  $\theta$  has one minimum, the position of which depends on  $\delta\omega$ .

The ground state is a coherent superposition of  $\varphi_0$  and  $\varphi_1$ . In this section, we confine our discussion in the first regime.

For  $g_{01} - \frac{1}{4}(g_0 + g_1) > 0$  and  $V_0 < g_{01} - \frac{1}{4}(g_0 + g_1)$ , there always exists  $\delta\omega = \frac{1}{4\hbar}(g_1 - g_0)$ , such that there exist two degenerate minima in the semiclassical energy. In this case, the true ground state of the system will be the superposition of the two macroscopic states due to the coherent quantum tunnelling between them which may be observed by interference experiments in principle.

As we have calculated explicitly, such conditions can be satisfied by a large class of systems such as Bose gas in an annulus or a thin cylindrical container. Tunnelling between the two degenerate macroscopic states leads to energy splitting between them. We evaluate this tunnelling splitting by applying the dilute instanton gas approximation which is valid for large- $N$ .

By adding some constants, the semiclassical energy can be rewritten as

$$E(\theta, \phi) = Ng(\sin \theta - \sin \theta_0)^2 + NV_0 \sin \theta (1 - \cos \phi), \quad (11)$$

where  $g \equiv \frac{1}{2}[g_{01} - \frac{1}{4}(g_0 + g_1)]$ ,  $\sin \theta_0 = V_0/2g$ . From  $\delta S_E(\theta, \phi) = 0$ , we get the instanton solution corresponding to the transition from  $\theta = \theta_0$  to  $\theta = \pi - \theta_0$ :

$$\begin{aligned} \bar{\theta} &= \arccos[-\cos \theta_0 \tanh(\omega_b \tau)] \\ \bar{\phi} &= \arcsin \left[ -\frac{i}{2} \frac{\cot^2 \theta_0 \text{sech}^2(\omega_b \tau)}{[1 + \cot^2 \theta_0 \text{sech}^2(\omega_b \tau)]^{1/2}} \right] \end{aligned} \quad (12)$$

where  $\omega_b = 2g \cos \theta_0 / \hbar$ . The associated instanton action is

$$S_{inst} = N \left[ -\cos \theta_0 + \frac{1}{2} \ln \left( \frac{1 + \cos \theta_0}{1 - \cos \theta_0} \right) \right], \quad (13)$$

and the tunnelling splitting is

$$\Delta E = 32g \left( \frac{N}{2\pi} \right)^{\frac{1}{2}} \frac{(\cos \theta_0)^{5/2}}{\sin \theta_0} \left( \frac{1 - \cos \theta_0}{1 + \cos \theta_0} \right)^{\frac{1}{2} \cos \theta_0} e^{-S_{inst}}. \quad (14)$$

#### IV. JOSEPHSON TUNNELLING

For  $V_0 > g_{01} - \frac{1}{4}(g_0 + g_1)$  and arbitrary  $\delta\omega$ , there always exists only one minimum in the semiclassical energy. In this section we evaluate the fluctuations for the relative number and phase around the ground state firstly. In the case where  $V_0 > g_{01} - \frac{1}{4}(g_0 + g_1)$  and

$\delta\omega = \frac{1}{4\hbar}(g_1 - g_0)$ , the classical ground state of the system is  $\bar{\theta} = \pi/2$ ,  $\bar{\phi} = 0$ . Expanding the Euclidean action  $S_E(\theta, \phi)$  to the second-order around its classical value  $\delta S_{cl} = 0$  and performing integrations over  $\phi_1(\tau) = \phi(\tau) - \bar{\phi}(\tau)$  and  $\theta_1(\tau) = \theta(\tau) - \bar{\theta}(\tau)$  respectively [4], we get the effective actions for  $\theta_1$  and  $\phi_1$  respectively

$$\begin{aligned} I(\theta_1) &= \int_{\tau_i}^{\tau_f} \left[ \frac{\hbar^2 N}{8V_0} \dot{\theta}_1^2 + \frac{N}{2}(V_0 + 2g)\theta_1^2 \right] d\tau, \\ I(\phi_1) &= \int_{\tau_i}^{\tau_f} \left[ \frac{\hbar^2 N}{8(V_0 + 2g)} \dot{\phi}_1^2 + \frac{V_0 N}{2}\phi_1^2 \right] d\tau. \end{aligned} \quad (15)$$

The motions of  $\theta_1$  and  $\phi_1$  are approximately harmonic oscillators and we get the relative number fluctuation

$$\Delta(N_0 - N_1) = \sqrt{N \sqrt{\frac{V_0}{V_0 - 2g}}}, \quad (16)$$

and the relative phase fluctuation

$$\Delta\phi = \sqrt{\frac{1}{N} \sqrt{\frac{V_0 - 2g}{V_0}}}. \quad (17)$$

In order to show the relationship between the N-body identical particle state and the condensate wave function, we rewrite the SU(2) coherent state  $|\theta, \phi\rangle$  as

$$|\theta, \phi\rangle = \frac{1}{\sqrt{N!}} (e^{-i\phi} \sin \frac{\theta}{2} a_1^+ + \cos \frac{\theta}{2} a_0^+)^N |0\rangle. \quad (18)$$

In coordinate representation, it corresponds to

$$\psi(\vec{r}_1, \dots, \vec{r}_N) = \prod_{r_i} \left( e^{-i\phi} \sin \frac{\theta}{2} \varphi_1(\vec{r}_i) + \cos \frac{\theta}{2} \varphi_0(\vec{r}_i) \right). \quad (19)$$

From the underlying concept of the off-diagonal-long-range-order (ODLRO) of C.N. Yang [5], we have

$$\int d^3 r_2 \dots d^3 r_N \psi^*(\vec{r}, \vec{r}_2, \dots, \vec{r}_N) \psi(\vec{r}', \vec{r}_2, \dots, \vec{r}_N) = \frac{1}{N} \Phi^*(\vec{r}) \Phi(\vec{r}'), \quad (20)$$

and  $\Phi(\vec{r}) = \sqrt{N} [e^{-i\phi} \sin \frac{\theta}{2} \varphi_1 + \cos \frac{\theta}{2} \varphi_0]$  is called the condensate wave function.

As we have calculated, for  $V_0$  sufficiently larger than  $2g$  we have  $\Delta(N_0 - N_1) \simeq \sqrt{N} \ll N$  and  $\Delta\phi \ll 1$ . This means that  $\varphi_1$  and  $\varphi_0$  can be viewed as almost two condensates and their phase difference is almost fixed for a large  $N$ . Thus, the ground state of the system satisfies the condition of the Josephson effect. It's easy to transfer the Hamiltonian (1) into a Josephson pendulum Hamiltonian

$$H = \frac{4g}{N} \frac{\partial^2}{\partial \phi^2} - V_0 N \sqrt{1 - \cos^2 \bar{\theta}} \cos \phi, \quad (21)$$

with the Josephson frequency  $\sqrt{8gV_0\sqrt{1-\cos^2\bar{\theta}}}$ .

It should be noted that, in the above we consider the case of  $\delta\omega = \frac{1}{4\hbar}(g_1 - g_0)$  in which the semi-classical ground state gives  $\cos\bar{\theta} = 0$ . The discussions above can be also applied to other cases in which  $\cos\bar{\theta} \neq 0$  and  $\cos\bar{\theta}$  changes gradually as  $\delta\omega - \frac{1}{4\hbar}(g_1 - g_0)$  increases until its absolute value reaches 1.

## V. NONLINEAR LANDAU-ZENER TUNNELLING

Consider a two-level system, its adiabatic eigenenergies  $\varepsilon_0(t)$  and  $\varepsilon_1(t)$  increases and decreases linearly with time  $t$  respectively, crossing at a certain value of  $t_0$ . Taking account of the coupling energy  $V_0$  between the two levels, the adiabatic eigenvalues of such a system  $E_0(t)$  and  $E_1(t)$  with wavefunctions  $\psi_0(t)$  and  $\psi_1(t)$  avoid crossing. We know from the adiabatic theorem that if the system with coupling is initially in the state  $\psi_0(t \ll t_0)$  and  $t$  changes infinitely slowly from  $t \ll t_0$  to  $t \gg t_0$ , then the system will remain in the state  $\psi_0(t \gg t_0)$ . However, if  $t$  changes with a finite velocity, the final state of the system will be a linear combination of  $\psi_0(t \gg t_0)$  and  $\psi_1(t \gg t_0)$ . This means when the two eigenenergies are getting close to each other, the system will undergo an instanton-like tunnelling, which is well known as Landau-Zener tunnelling [6] and can be realized in a lot of systems in molecular and atomic physics [7]. The essential idea of the Landau-Zener tunnelling can be generalized to the many-body mean-field problem with non-linear interactions [2], in which the tunnelling can take place even in the adiabatic case.

In the two previous sections we study the case where the angular frequency  $\omega$  is fixed while it can be dependent of time in real life. Thus, it is necessary to consider the case where  $\omega$  varies with time, accompanied by a non-linear Landau-Zener tunnelling.

The equations of motion of the operators  $a_0$  and  $a_1$  can be derived from the Hamiltonian (1)

$$\begin{aligned} i\hbar \frac{da_0}{dt} &= [\hbar\delta\omega + U(\tilde{g}_0 a_0^\dagger a_0 + 2\tilde{g}_{01} a_1^\dagger a_1)]a_0 - V_0 a_1 \\ i\hbar \frac{da_1}{dt} &= [-\hbar\delta\omega + U(\tilde{g}_1 a_1^\dagger a_1 + 2\tilde{g}_{01} a_0^\dagger a_0)]a_1 - V_0 a_0 \end{aligned} \quad (22)$$

Applying the mean-field approximation while dropping the fluctuations of the field operators

$$\langle a_1 \rangle = \langle a_1^\dagger \rangle^* = x_1 \sqrt{N}$$

$$\langle a_0 \rangle = \langle a_0^+ \rangle^* = x_0 \sqrt{N} , \quad (23)$$

we get a non-linear Landau-Zener equation

$$i\hbar \frac{d}{dt} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \hbar\delta\omega + d_1 + c_1(|x_1|^2 - |x_0|^2) & -V_0 \\ -V_0 & -\hbar\delta\omega + d_2 - c_2(|x_1|^2 - |x_0|^2) \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \quad (24)$$

where

$$\begin{aligned} d_1 &= \frac{1}{2}(g_0 + 2g_{01}) , \\ c_1 &= \frac{1}{2}(g_0 - 2g_{01}) , \\ d_2 &= \frac{1}{2}(g_1 + 2g_{01}) , \\ c_2 &= \frac{1}{2}(g_1 - 2g_{01}) , \end{aligned} \quad (25)$$

$|x_0|^2, |x_1|^2$  are the probabilities in corresponding states and  $|x_0|^2 + |x_1|^2 = 1$ . Our model is a more general one than that of Ref. [2] for the influence of the interaction on these two modes is different. If  $\hbar\delta\omega$  varies linearly with time, i.e.  $\hbar\delta\omega = \alpha t$ , we know from the linear Landau-Zener effect that an instanton-like transition will occur at times near to the crossing point of the unperturbed energy levels. As the role of the parameters  $d_1$  and  $d_2$  in Eq. (24) is to shift the crossing point of the unperturbed energy levels, we have reason to expect the dimensionless parameter  $G \equiv (g_0 - g_1)/g_0$  will only affect the transition time. Another characteristic dimensionless parameter of the system  $K \equiv (g_1 - 2g_{01})/(g_0 - 2g_{01})$  is given by the imbalance of the particle interactions.

The system of non-linear equations is solved numerically. Fig. 1 shows that the transition probability as a function of  $K$  for  $t \rightarrow +\infty$  is nearly monotonic increasing. The plots with different values of the parameter  $G$  are almost identical, which implies the tunnelling probabilities are not sensitive to  $G$  as we have predicted. As we can see, the system is in a coherent state of the two states  $\varphi_0$  and  $\varphi_1$  because of the nonzero transition probabilities even in the adiabatic case. This can also explain the hysteresis phenomenon of the vortex formation.



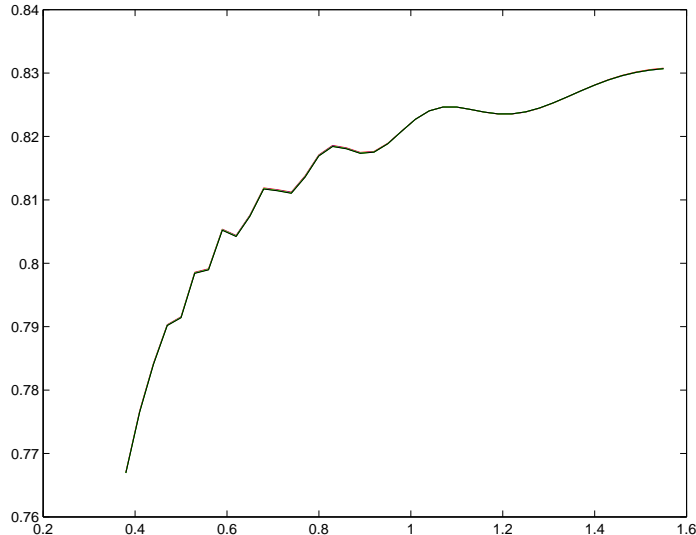


FIG. 1: Tunnelling probability as a function of the dimensionless parameter  $K$  is plotted for different values (0.1, 0.2, 0.3, 0.4, 0.5) of the parameter  $G$ . The plots for different values of the parameter  $G$  are almost identical to one another.

## VI. SUMMARY AND CONCLUSIONS

In summary, we have studied the macroscopic phenomena in rotating BEC by using the SU(2) coherent state path-integral method. Our discussion can also be extended to the rotating BEC with attractive interactions and BEC in an optical lattice. In the latter case, we can expand the field operator in terms of single-particle ground state wave functions localized in each well, then the coupling energy  $V_0$  is replaced by the hopping energy, which depends on the overlap of the local wave functions. Some detailed calculation can be found in an earlier paper [4]. We can see that when  $V_0$  is sufficiently small, the phase fluctuation will be very large,  $\sim 2\pi$ . According to Eq. (6), the system is in a Fock region. It can be easily understood physically in the rotating case that when the coupling energy is absent, the tunnelling is forbidden by the angular momentum conservation law and the relative number fluctuation is suppressed. In an optical lattice, small  $V_0$  means sufficiently high barrier which also prohibits the tunnelling. When  $V_0$  is comparable to the self-interaction energy, the system is in the Josephson region as we have shown in section IV. Thus, continuously decreasing  $V_0$  will lead the system to undergo a phase-diffusion process and there is a crossover from the Josephson to Fock region as exhibited experimentally recently in Ref.

[8]. The physics of such experiments can be understood clearly and quantitatively in our SU(2) coherent state frame.

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- [1] Anthony J. Leggett, Rev. Mod. Phys. **73**, 307 (2001) and references therein.
- [2] Biao Wu and Qian Niu, Phys. Rev. A **61**, 023402 (2000).
- [3] Wei-Min Zhang, Da Hsuan Feng and Robert Gilmore, Rev. Mod. Phys. **62**, 867 (1990).
- [4] Yi Zhou, Hui Zhai, Rong Lu, Zhan Xu, Lee Chang, cond-mat/0104361 .
- [5] C.N.Yang, Rev. Mod. Phys. **34**, 4 (1962).
- [6] L.D. Landau, Phys. Z. Sowjetunion **2**, 46 (1932); G. Zener, Proc. R. Soc. London, Ser. A **137**, 696 (1932).
- [7] W. Wernsdorfer and R. Sessoli, Science **284**, 134 (1999); K. Mullen *et al.*, Phys. Rev. Lett. **60**, 1097 (1988); C.F. Bharucha *et al.*, Phys. Rev. A **55**, R857 (1997); A. Sibille *et al.*, Phys. Rev. Lett. **80**, 4506 (1998).
- [8] Markus Greiner, Olaf Mandel, Tilman Esslinger, Theodor W. Hänsch and Immanuel Bloch, Nature **415**, 39 (2002).